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On the Rate of Almost Everywhere Convergence of Certain Classical Integral Means, II

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In this sequel to previous work of A. Stokolos and W. Trebels (1999, *J. Approx. Theory* **98**, 203–222) we indicate at the example of the Gauss–Weierstrass and the Abel–Poisson means the sharpness of some results obtained there. This is achieved by modifying methods of K. I. Oskolkov (1977, *Math. USSR-Sb.* **32**, 489–514) and A. A. Soljanik (1986, Ph.D. Thesis, Odessa) developed for the periodic case. © 1999 Academic Press

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1. INTRODUCTION AND MAIN RESULTS

In [11] some contributions are given to the following problem. Suppose that a function f has a certain smoothness property in $L^p(\mathbf{R})$-norm. What is a good/natural rate of convergence by which certain approximation processes of convolution type converge a.e. towards f? In this sequel to [11] we discuss the sharpness of the results obtained there. We modify methods

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used by Oskolkov [6, 7] in a similar circle of problems and by Soljanik [9] in the periodic case; a substantial role is played by a lemma due to Calderón (see, e.g., [10, p. 442]). The same method also gives the sharpness of Soljanik's [9] results in the Hardy classes $H^p(\mathbf{R}^2_+)$, $1 \leqslant p < \infty$.

First introduce the L^p-modulus of continuity of a function $f \in L^p(\mathbf{R})$, $1 \leqslant p < \infty$, by

$$\omega(f, t)_p = \sup_{|h| < t} \|\Delta_h f\|_{L^p(\mathbf{R})}, \qquad \Delta_h f(x) = f(x+h) - f(x).$$

Further, consider continuous increasing sub-additive functions $\omega(t)$ on $(0, \infty)$ with $\lim_{t \to 0+} \omega(t) = 0$; define classes of smoothness $H^\omega_p(\mathbf{R})$ by

$$H^\omega_p(\mathbf{R}) = \{ f \in L^p(\mathbf{R}) \colon \omega(f, t)_p \leqslant C\omega(t) \}.$$

By $W_t(f)$ we denote the Gauss–Weierstrass means (observe the different normalization of the parameter t), defined on $L^2(\mathbf{R})$ by

$$W_t(f)(x) = \int_{\mathbf{R}} e^{-t^2 |\xi|^2} \hat{f}(\xi)\, e^{ix\xi}\, d\xi, \qquad \hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} f(x)\, e^{-i\xi x}\, dx.$$

Corollary A (see [11]). *If $f \in H^\omega_p(\mathbf{R})$, $1 \leqslant p < \infty$, and $\varepsilon > 0$ is arbitrary then, for $t \to 0+$ and almost all $x \in \mathbf{R}$, there holds*

$$W_t(f)(x) - f(x)$$
$$= o_x(\omega(t)) \begin{cases} (\log \log(1/t))^{1/p+\varepsilon}, & \omega(t) = t(\log(1/t))^\lambda, \quad \lambda > 0 \\ \left(\log \dfrac{1}{t}\right)^{1/p+\varepsilon}, & \omega(t) = t^\lambda, \quad 0 < \lambda < 1 \\ (\log \log(1/t))^{1/p+\varepsilon}, & \omega(t) = \left(\log \dfrac{1}{t}\right)^{-\lambda}, \quad \lambda > 0. \end{cases}$$

Part (b) of Theorem 1 below will show that it is impossible to choose $\varepsilon = 0$ in Corollary A. Our main result deals with approximation processes $(T_{m_t})_{t>0}$ of convolution type

$$T_{m_t} f(x) = \int_{\mathbf{R}} m(t\,|\xi|)\, \hat{f}(\xi)\, e^{ix\xi}\, d\xi, \qquad f \in L^2(\mathbf{R}), \tag{1}$$

where $m \in L^\infty(0, \infty)$. So the Gauss–Weierstrass integral is generated by $m(u) = e^{-u^2}$.

THEOREM 1. (i) *Let* $\Phi \in L^1(\mathbf{R})$, $\hat{\Phi}(\xi) = m(|\xi|)$, *be an even kernel with*

$$\int_{\mathbf{R}} \Phi(x)\, dx = 1, \qquad \int_{\mathbf{R}} |x\Phi(x)|\, dx < \infty,$$

the radial majorant $\bar{\Phi}(x) \equiv \sup_{|y| \geqslant |x|} |\Phi(y)|$ *belonging to* $L(\mathbf{R})$.

(ii) *Let* $\omega(t)$ *be a modulus of continuity such that* $\omega(t)/t \uparrow \infty$, $t \to 0+$. *Define* δ_k *in the following way*

$$\delta_0 = 1, \qquad \delta_{k+1} = \min\left\{\delta : \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}; \frac{\delta\omega(\delta_k)}{\delta_k\,\omega(\delta)}\right) = \frac{1}{2}\right\}, \quad k = 0, 1, ... \tag{2}$$

(iii) *Let* $w(t)$ *be a nondecreasing positive function such that* $\omega(t)/w(t)$ *is nondecreasing.*

(a) (*See* [11, *Theorem* 2.1].) *If*

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)}\right)^p < \infty, \tag{3}$$

then, for every function $f \in H_p^\omega(\mathbf{R})$, $1 \leqslant p < \infty$, *there holds*

$$T_{m_t} f(x) - f(x) = o_x(w(t)) \qquad \text{a.e.}, \quad t \to 0+. \tag{4}$$

(b) *If the series in* (3) *diverges, i.e., if*

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)}\right)^p = \infty, \tag{5}$$

and if additionally $\Phi \geqslant 0$, *then there exists an* $f \in H_p^\omega(\mathbf{R})$, $1 \leqslant p < \infty$, *such that*

$$\limsup_{t \to 0+} \frac{|T_{m_t} f(x) - f(x)|}{w(t)} = \infty \qquad \text{a.e.} \tag{6}$$

Remark 1. Theorem 1 is in the spirit of results due to K. I. Oskolkov [6] concerning Steklov means of periodic functions. We recall the following crucial property of the Oskolkov sequence $\{\delta_k\}$ (see [6; 11, Lemma E])

$$C^{-1}\,\omega(\delta) \leqslant \sum_{k=0}^{\infty} \omega(\delta_k) \min\left\{1, \frac{\delta}{\delta_k}\right\} \leqslant C\omega(\delta), \qquad \delta \in (0, 1], \tag{7}$$

(for some absolute constant $0 < C < \infty$) which in fact can be taken as definition, while (2) is an explicit realization of such sequences. This choice of $\{\delta_k\}$ allows to handle smoothness near the optimal $\omega^*(t) = t$ in contrast to the $\{\theta_k\}$'s from Theorem 3 and Corollary 5 below.

It is interesting to discuss in how far the hypothesis $\bar{\Phi}(x) \equiv \sup_{|y| \geqslant |x|} |\Phi(y)| \in L^1(\mathbf{R})$ can be modified—see in this respect, e.g., [5, Chap. 10].

If one compares the two parts of Theorem 1 it becomes clear that for positive approximation processes condition (3) cannot be weakened in this setting. Let us look at the examples given in [11, Corollary 2.1.1]: By a result of B. Kuttner (see [4]), the Riesz typical means generated by $m(u) = (1-u^\gamma)_+^\alpha$ are positive means provided $0 < \gamma < 2$ and $\alpha \geqslant a(\gamma)$ where $a(\gamma)$ is a strictly increasing function, continuous on $(0, 2)$ with $0 < a(0+) < 1$, $a(1) = 1$, and $a(2-) = \infty$. Thus, if $\alpha > 1$ and $1 < \gamma < 2$, Theorem 1 applies (see [11, Corollary 2.1.1]). Also, due to B. Kuttner, the Abel–Cartwright means of order γ, $0 < \gamma \leqslant 2$, generated by $m(u) = e^{-u^\gamma}$, are positive so that for $\gamma > 1$ we have another example for Theorem 1. In particular, we can state the following corollary.

COROLLARY 1. *One cannot improve the estimates in Corollary* A *by choosing* $\varepsilon = 0$.

The Abel–Poisson means $(P_t f)_{t>0}$,

$$P_t(f)(x) = \int_{\mathbf{R}} e^{-t|\xi|} \hat{f}(\xi)\, e^{ix\xi}\, d\xi, \qquad f \in L^2(\mathbf{R}),$$

though positive, do not fall under the scope of Theorem 1. But some results concerning the pointwise approximation behavior of P_t can be obtained from the theory of Hardy spaces $H^p(\mathbf{R}^2_+)$, $1 \leqslant p < \infty$. These are defined as the set of functions $F(z)$ holomorphic in the upper half-plane with

$$\|F\|^p_{H^p(\mathbf{R}^2_+)} \equiv \sup_{y>0} \int_{\mathbf{R}} |F(x+iy)|^p\, dx < \infty.$$

It is well known [10, p. 127] that the boundary values $\lim_{y \to 0+} F(x+iy) = F(x)$ exist almost everywhere for elements from $H^p(\mathbf{R}^2_+)$, belong to $L^p(\mathbf{R})$ and that $\|F\|_p = \|F\|_{H^p(\mathbf{R}^2_+)}$. As modulus of continuity of $F(z) \in H^p(\mathbf{R}^2_+)$ we define the modulus of continuity of its boundary value $F(x)$, further the smoothness classes $H^\omega_p(\mathbf{R}^2_+)$ by

$$H^\omega_p(\mathbf{R}^2_+) = \{F(z) \in H^p(\mathbf{R}^2_+) \colon \omega(F, t)_p \leqslant C\omega(t)\}.$$

THEOREM 2. *Assume that the hypotheses* (ii) *and* (iii) *of Theorem* 1 *hold.*

(a) (*See A. A. Soljanik* [9].) *If* (3) *is satisfied then, for every* $F \in H_p^\omega(\mathbf{R}_+^2)$, $1 \leqslant p < \infty$, *we have*

$$F(x+it) - F(x) = o_x(w(t)) \qquad a.e., \quad t \to 0+. \tag{8}$$

(b) *If* (5) *holds, then there exists an* $F \in H_p^\omega(\mathbf{R}_+^2)$, $1 \leqslant p < \infty$, *such that*

$$\limsup_{t \to 0+} \frac{|F(x+it) - F(x)|}{w(t)} = \infty \qquad a.e. \tag{9}$$

Remark 2. A. A. Soljanik [9] has in fact shown that Part (a) of Theorem 2 is true for all $p > 0$. Theorem 2 implies sharp estimates for the Abel–Poisson means: If $f \in L^p(\mathbf{R})$, $1 \leqslant p < \infty$, then we have that the function $F = f + iHf$ (where Hf denotes the Hilbert transform of f) is the boundary value of some function from $H^p(\mathbf{R}_+^2)$ if and only if $Hf \in L^p(\mathbf{R})$. In this case $F(x+iy) = P_y(F)(x)$. Furthermore, if $1 < p < \infty$, then the Hilbert transform is a bounded operator on $L^p(\mathbf{R})$, which implies that $C^{-1}\omega(F, t)_p \leqslant \omega(f, t)_p \leqslant C\omega(F, t)_p$ for some constant $0 < C < \infty$ independent of t. Thus the set of boundary values of analytic functions belonging to $H_p^\omega(\mathbf{R}_+^2)$, $1 < p < \infty$, coincides with $H_p^\omega(\mathbf{R})$ and Theorem 2 has the following corollaries (for the direct estimates see [11]).

COROLLARY 2. *Let* $1 < p < \infty$ *and* $w(t)$ *be a nondecreasing positive function such that* $\omega(t)/w(t)$ *is nondecreasing and* (3) *is true. Then, for every function* $f \in H_p^\omega(\mathbf{R})$ *there holds*

$$P_t(f)(x) - f(x) = o_x(w(t)) \qquad a.e., \quad t \to 0+. \tag{10}$$

The estimate is sharp in the sense of Theorem 2.

COROLLARY 3. *If* $f \in H_p^\omega(\mathbf{R})$, $1 < p < \infty$, *if* $\omega(t)\, t^{-s}$, $s > 0$, *is an increasing function with*

$$\int_0^1 \left(\frac{\omega(t)}{t^s}\right)^p \frac{dt}{t} < \infty \tag{11}$$

then, for almost all $x \in \mathbf{R}$,

$$P_t f(x) - f(x) = o_x(t^s), \qquad t \to 0+.$$

If (11) *does not hold then there exists a function* $f \in H_p^\omega(\mathbf{R})$ *such that*

$$\limsup_{t \to 0+} |P_t f(x) - f(x)|\, t^{-s} = \infty \qquad a.e.$$

In the case $p=1$ the function $F(x)=f(x)+iHf(x)$ is not necessarily integrable. Nevertheless, for the bulk of L^1-functions, in a certain sense "modestly" smooth, we have that $\Delta_h Hf \in L^1(\mathbf{R})$ and that the modulus of continuity of Hf can be estimated by that of f (see [2, p. 61]), namely

$$\omega(Hf; u)_1 \leqslant C\left(\int_0^u \frac{\omega(f, t)_1}{t}\,dt + u\int_u^\infty \frac{\omega(f, t)_1}{t^2}\,dt\right). \tag{12}$$

Obviously, this implies the corresponding estimate for $\omega(F, t)_1$ by $\omega(f, t)_1$.

Corollary 4. *Let $\omega(t)$ be a modulus of continuity with $\omega(t)/t \uparrow \infty$, $t \to 0+$.*

(a) *Assume that*

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty,$$

that $\tilde{\omega}(t)$ is a further modulus of continuity satisfying

$$\int_0^u \frac{\omega(t)}{t}\,dt + u\int_u^\infty \frac{\omega(t)}{t^2}\,dt \leqslant C\tilde{\omega}(u),$$

that $w(t)$ is a nondecreasing positive function such that $\tilde{\omega}(t)/w(t)$ is nondecreasing, and

$$\sum_{k=1}^\infty \frac{\tilde{\omega}(\delta_k)}{w(\delta_k)} < \infty.$$

Then, for all $f \in H_1^\omega(\mathbf{R})$, we have

$$P_t f(x) - f(x) = o_x(w(t)) \qquad \text{a.e.,} \quad t \to 0+.$$

(b) *Suppose that $\omega(t)$ is a bounded function such that $\omega(t)/t$ is decreasing and that*

$$\lim_{t \to 0+} \frac{\omega(2t)}{\omega(t)} = 2. \tag{13}$$

Suppose further that $w(t)$ satisfies condition (iii) *of Theorem* 1 *and* (3) *then, for all $f \in H_1^\omega(\mathbf{R})$, we have*

$$P_t f(x) - f(x) = o_x(w(t)\log(1/t)) \qquad \text{a.e.,} \quad t \to 0+.$$

The assumption ω being bounded implies in practice of course no loss of generality since for any $f \in L^1(\mathbf{R})$ trivially $\omega(f; t)_1 \leqslant 2\,\|f\|_1$. Condition (13) means that ω is near to the best possible case $\omega^*(t)=t$. To illustrate this, consider $\omega(t)=t(\log(1/t))^\alpha$, $0<t\leqslant 1/2$, $\alpha>0$, ω extended appropriately to $t>1/2$. Then, for every $f\in H_1^\omega(\mathbf{R})$, there holds

$$P_t(f)(x)-f(x)=o_x(t(\log(1/t))^{\alpha+1}(\log\log(1/t))^{1+\varepsilon}) \qquad \text{a.e.,} \quad t\to 0+.$$

Corollary 4, Part (a) immediately follows from Theorem 2. To realize that this is also the case for Part (b) we note the two estimates: (i) that by the hypothesis (13)

$$\int_0^{2^{-n}} \omega(t)/t\,dt\leqslant C\sum_{k=n}^{\infty}\omega(2^{-k})\leqslant C\omega(2^{-n})$$

therefore,

$$\int_0^u \omega(t)/t\,dt\leqslant C\omega(u),$$

and (ii), since $\omega(t)/t$ is assumed to be decreasing, that

$$u\int_u^\infty \omega(t)/t^2\,dt\leqslant u\int_u^1 \omega(t)/t^2\,dt+Cu\int_1^\infty t^{-2}\,dt$$

$$\leqslant Cu\omega(u)/u\int_u^1 1/t\,dt+Cu\leqslant C\omega(u)\log(1/u).$$

Let us note, that Corollary 4 does not cover the case of "bad" moduli, e.g., those of logarithmic type. This type falls under the scope of the following

THEOREM 3. (i) *Let $\omega(t)$ be a modulus of continuity which satisfies the doubling condition*

$$\limsup_{t\to 0+}\frac{\omega(2t)}{\omega(t)}<2.$$

(ii) *Let $w(t)$ be a nondecreasing function such that $\omega(t)/w(t)$ is nondecreasing.*

(a) (*see* [11].) *If θ_k is defined by $\omega(\theta_k)=2^{-k}$ and if*

$$\sum_{k=1}^{\infty}\frac{\omega(\theta_k)}{w(\theta_k)}<\infty, \tag{14}$$

then, for all $f \in H_1^{\omega}(\mathbf{R})$, *there holds*

$$P_t f(x) - f(x) = o_x(w(t)) \qquad a.e., \quad t \to 0+. \tag{15}$$

(b) *If* θ_k *is defined as in* (a) *and if the series in* (14) *diverges, i.e., if*

$$\sum_{k=1}^{\infty} \frac{\omega(\theta_k)}{w(\theta_k)} = \infty, \tag{16}$$

then there exists a function $f \in H_1^{\omega}(\mathbf{R})$ *with*

$$\limsup_{t \to 0+} \frac{|P_t f(x) - f(x)|}{w(t)} = \infty \qquad a.e. \tag{17}$$

Theorem 3 yields the following corollary (for the positive part see [11]).

COROLLARY 5. *If* $f \in H_1^{\omega}(\mathbf{R})$ *and* $\varepsilon > 0$ *is arbitrary then, for* $t \to 0+$ *and almost all* $x \in \mathbf{R}$,

$$P_t(f)(x) - f(x)$$
$$= o_x(\omega(t)) \begin{cases} \left(\log \frac{1}{t}\right)^{1+\varepsilon}, & \omega(t) = t^{\lambda}, \quad 0 < \lambda < 1 \\ (\log \log(1/t))^{1+\varepsilon}, & \omega(t) = \left(\log \frac{1}{t}\right)^{-\lambda}, \quad \lambda > 0. \end{cases}$$

This result is sharp in so far as it does not remain true for $\varepsilon = 0$.

There is only to prove Theorems 1, 2, 3 which is done in Sections 2, 3, 4.

2. PROOF OF THEOREM 1

Proof. By the results in [11] there is only to show Part (b). To this end we modify Oskolkov's approach [6] for the Steklov means on the one-dimensional torus appropriately. We construct a function f as the sum of simple nonnegative polygonal functions f_k, $f = \sum_{k \in \mathscr{L}} f_k$. By a lemma of Calderón on shifts (see, e.g., [10, p. 442]) we can distribute the f_k's on the real line such that, in fact, almost all points x are contained in infinitely many supports of the f_k's. Additionally we distribute the f_k's in such a way, that for almost all x the sets of those indexes for which $f_k(x) \neq 0$ are quite thin (see (28)). The estimates concerning the smoothness of f are not difficult due to the remarkable properties of the sequence (2).

For the estimate of the difference $T_{m_t} f(x) - f(x)$, for fixed x, we select one term, say the one corresponding to f_s, which gives the main contribution. The contributions corresponding to the f_k's, $k < s$, are then relatively small due to the small parameter in the approximation and the rest of the series gives a small contribution due to the lacunarity of the L^∞ norms of the elements of series.

We first assume that the kernel Φ is decreasing on $[0, \infty)$ and will discard of this additional assumption later. Further we note two simplifications.

(i) It is sufficient to prove the existence of some $f \in H_p^\omega(\mathbf{R})$ with

$$\limsup_{t \to 0+} \frac{|T_{m_t} f(x) - f(x)|}{w(t)} > 0 \qquad \text{a.e.} \quad \text{on } \mathbf{R} \tag{18}$$

instead of (6). For if $\tilde{w}(t)$ satisfies (5), we may choose $w(t)$ such that (5) holds and additionally $w(t)/\tilde{w}(t) \uparrow \infty$, $t \to 0+$, thus by (18)

$$\limsup_{t \to 0+} \frac{|T_{m_t} f(x) - f(x)|}{\tilde{w}(t)}$$

$$= \limsup_{t \to 0+} \frac{|T_{m_t} f(x) - f(x)|}{w(t)} \frac{w(t)}{\tilde{w}(t)} = +\infty \qquad \text{a.e. on } \mathbf{R}.$$

(ii) Without loss of generality we may assume that

$$\psi_1 = 1, \qquad \psi_k \geqslant k, \quad \psi_k := \left(\frac{w(\delta_k)}{\omega(\delta_k)} \right)^p. \tag{19}$$

For, by hypothesis, $\{1/\psi_k\}$ is a non-increasing sequence with

$$\sum_{k=1}^{\infty} \frac{1}{\psi_k} = \infty. \tag{20}$$

Now (ii) holds if we can show that

$$\sum_{k=1}^{\infty} \min\left(\frac{1}{k}; \frac{1}{\psi_k} \right) = \infty. \tag{21}$$

Indeed, if $1/\psi_k = o(1/k)$ then (21) follows immediately from (20). Otherwise there exists a lacunary sequence $k_i \uparrow \infty$, $k_{i+1} \geqslant 2k_i$ such that $1/\psi_{k_i} \geqslant \alpha/k_i$ with some positive $\alpha \leqslant 1$. Since the sequence $\{\psi_k\}$ is monotone we obtain

$$\frac{1}{\psi_j} \geqslant \frac{\alpha}{2} \frac{1}{j}, \qquad \frac{k_i}{2} \leqslant j \leqslant k_i,$$

in particular, $\min(1/j, 1/\psi_j) \geqslant \alpha/2j$ for these j, whence

$$\sum_{k=1}^{\infty} \min\left(\frac{1}{k}; \frac{1}{\psi_k}\right) \geqslant \frac{\alpha}{2} \sum_{i=1}^{\infty} \sum_{k_i/2 < j \leqslant k_i} \frac{1}{j} \geqslant \frac{\alpha}{2} \sum_{i=1}^{\infty} \frac{1}{k_i} \cdot \frac{k_i}{2} = \infty.$$

With the numbers $\{\delta_k\}$ as in (2) we now define

$$r_k = \max\{m \in \mathbf{Z}_+ : 2m\delta_k \leqslant 1/\psi_k\}, \qquad k = 1, 2, ..., \tag{22}$$

and consider the subset $\mathscr{K}$ of $\mathbf{Z}_+$,

$$\mathscr{K} = \{k : \psi_k \leqslant k^2\}. \tag{23}$$

It follows from (20) that

$$\sum_{k \in \mathscr{K}} r_k \, \delta_k = \infty. \tag{24}$$

Let $F_k = (a_k; b_k]$ be some intervals such that

$$|F_k| = b_k - a_k = 2r_k \, \delta_k.$$

Put $a_2 = 0$ and $a_{k+1} = b_k$ if $b_k < 1$ and $a_{k+1} = 0$ otherwise. Thus we have correctly defined the intervals F_k for $k \geqslant 2$. Let $s_m \uparrow \infty$ be such that $a_{s_m} = 0$ and consider

$$E_k = \bigcup_{v=1}^{r_k - 1} \left[a_k + v 2\delta_k - \frac{\Phi(1)}{2} \delta_k; a_k + v 2 \, \delta_k + \frac{\Phi(1)}{2} \delta_k \right].$$

Then

$$|E_k| = 2(r_k - 1) \frac{\Phi(1)}{2} \delta_k \geqslant C r_k \, \delta_k, \qquad \sum_{k \in \mathscr{K}} |E_k| = \infty$$

by (24). Let

$$\mathscr{L} = \bigcup_{m=1}^{\infty} \mathscr{L}_m, \qquad \mathscr{L}_m \equiv \{k \in \mathscr{K} : s_{2m} \leqslant k < s_{2m+1}\}, \quad E_m^* = \bigcup_{k \in \mathscr{L}_m} E_k. \tag{25}$$

Then obviously

$$\sum_{k \in \mathscr{L}} |E_k| = \infty \tag{26}$$

or $\sum_{k \notin \mathscr{L}} |E_k| = \infty$. Without loss of generality assume (26) and rewrite it as

$$\sum_{m=1}^{\infty} |E_m^*| = \sum_{k \in \mathscr{L}} |E_k| = \infty.$$

By the Calderón lemma (see, e.g., [10, p. 442]), there exist numbers ξ_m such that

$$\limsup_m E_{\xi_m}^* \cup E \equiv \left(\bigcap_{j=1}^{\infty} \bigcup_{m=j}^{\infty} E_{\xi_m}^* \right) \cup E = \mathbf{R}, \tag{27}$$

where $E_{\xi_m}^* = E_m^* - \xi_m$ are translates of E_m^* and E is some set of measure zero. Denote by τ_m the translation $\tau_m(\cdot) \equiv (\cdot - \xi_m)$ and define

$$F_k^\tau \equiv \tau_m(F_k) = (\alpha_k; \beta_k], \qquad s_{2m} \leqslant k < s_{2m+1}.$$

Further, for $x \in \mathbf{R}$ introduce a subset of $\mathscr{L}$ by $\mathscr{K}_x = \{k \in \mathscr{L} : F_k^\tau \ni x\}$. By the choice of F_k^τ, if $k \in \mathscr{K}_x$ and $l \in \mathscr{K}_x$, $l > k$, there follows

$$s_{2j} \leqslant k < s_{2j+1} < s_{2m} \leqslant l < s_{2m+1}$$

for some j and m which implies that $1 \leqslant \sum_{s=k}^{l} 2r_s \delta_s$. Then by (19)

$$1 \leqslant \sum_{s=k}^{l} 2r_s \delta_s \leqslant \sum_{s=k}^{l} \psi_s^{-1} \leqslant \sum_{s=k}^{l} 1/s \leqslant \int_{1/(l-1)}^{1/k} \frac{dx}{x} = \ln \frac{l-1}{k}.$$

Thus we have established the following important property of $\mathscr{K}_x$: there exists a $k_0 \geqslant 1$ such that for any $x \in \mathbf{R}$

$$l, k \in \mathscr{K}_x, \qquad l > k, \quad \text{implies} \quad l \geqslant 2k, \quad k \geqslant k_0. \tag{28}$$

Define

$$\mathscr{I}_k := \{x : x = \alpha_k + s2\delta_k,\ s = 0, 1, ..., r_k\} \subset F_k^\tau,$$

polygonal functions

$$f_k(x) := \psi_k^{1/p} \frac{\omega(\delta_k)}{\delta_k} \begin{cases} \operatorname{dist}(x, \mathscr{I}_k) & \text{if } x \in F_k^\tau \\ 0, & \text{if } x \notin F_k^\tau, \end{cases}$$

and finally f by $f = \sum_{k \in \mathscr{L}} f_k$.

We show that $f \in H_p^\omega(\mathbf{R})$. It is clear that the f_k's are absolutely continuous functions with

$$f_k(x) = 0 \qquad \text{if} \quad x \notin F_k^\tau, \qquad \sup_{x \in F_k^\tau} |f_k(x)| \leqslant \omega(\delta_k)\, \psi_k^{1/p}, \tag{29}$$

and

$$f'_k(x)=0 \qquad \text{if} \quad x \notin F_k^\tau, \qquad \sup_{x\in F_k^\tau} |f'_k(x)| \leqslant \frac{\omega(\delta_k)}{\delta_k} \psi_k^{1/p}. \tag{30}$$

It follows from (29) and (30) that

$$\omega(f_k;\delta)_\infty \leqslant \psi_k^{1/p} \min\left(\frac{\omega(\delta_k)}{\delta_k}\,\delta,\, \omega(\delta_k)\right).$$

Furthermore, (see (22))

$$\omega(f_k;\delta)_p \leqslant \omega(f_k;\delta)_\infty\, (2\,|F_k^\tau|)^{1/p} = \omega(f_k;\delta)_\infty\, (4r_k\delta_k)^{1/p}$$

$$\leqslant 2\min\left(\frac{\omega(\delta_k)}{\delta_k}\,\delta,\, \omega(\delta_k)\right).$$

Since Oskolkov's sequence $\{\delta_k\}$ has the property (7) we have $f \in H_p^\omega(\mathbf{R})$. We come to the estimate of $T_{m_t} f(x) - f(x)$. Take $t=\delta_s$ with $s \in \mathscr{L}$; set $\mathscr{K}_x^c = \mathscr{L} \setminus \mathscr{K}_x$. Then

$$\begin{aligned} T_{m_t} f(x) - f(x) = &\sum_{k<s,\, k\in \mathscr{K}_x^c} T_{m_t} f_k(x) + \sum_{k<s,\, k\in\mathscr{K}_x} (T_{m_t} f_k(x) - f_k(x)) \\ &+ (T_{m_t} f_s(x) - f_s(x)) \\ &+ \sum_{k>s,\, k\in\mathscr{K}_x} (T_{m_t} f_k(x) - f_k(x)) \\ &+ \sum_{k>s,\, k\in\mathscr{K}_x^c} T_{m_t} f_k(x). \end{aligned}$$

Since the first and the last sum of the right hand side are nonnegative, there holds

$$T_{m_t} f(x) - f(x) \geqslant II(x) - |I(x)| - |III(x)|,$$

where

$$\begin{aligned} I(x) &\equiv \sum_{k<s,\, k\in\mathscr{K}_x} (T_{m_t} f_k(x) - f_k(x)), \\ II(x) &\equiv T_{m_t} f_s(x) - f_s(x), \\ III(x) &\equiv \sum_{k>s,\, k\in\mathscr{K}_x} (T_{m_t} f_k(x) - f_k(x)). \end{aligned}$$

We first show that

$$I(x)+III(x)=o(w(t)) \qquad \text{a.e. on } \mathbf{R} \tag{31}$$

and then

$$II(x)\geqslant C_x w(t) \qquad \text{a.e. on } \mathbf{R}. \tag{32}$$

We start with the estimate of $III(x)$. By the properties (23), (29), and (28) for large enough s

$$\begin{aligned}
|III(x)| &\leqslant \sum_{k>s,\, k\in\mathscr{K}_x} \int_R |f_k(x+h)-f_k(x)|\, \Phi_t(h)\, dh \\
&\leqslant \sum_{k>s,\, k\in\mathscr{K}_x} 2\omega(\delta_k)\, \psi_k^{1/p} \leqslant C \sum_{k>s,\, k\in\mathscr{K}_x} k^2\omega(\delta_k) \\
&\leqslant C\omega(\delta_{s+1}) \sum_{k\in\mathscr{K}_x,\, k>s} k^2 2^{s-k+1} \\
&\leqslant C\omega(\delta_{s+1}) \sum_{k>2s} k^2 2^{k-s+1} = o(\omega(\delta_s)).
\end{aligned}$$

Next consider $I(x)$. By (23), (30), and hypothesis (i)

$$\begin{aligned}
|I(x)| &\leqslant \sum_{k<s,\, k\in\mathscr{K}_x} \int_R |f_k(x+h)-f_k(x)|\, \Phi_t(h)\, dh \\
&\leqslant \sum_{k=2}^{s/2} \frac{\omega(\delta_k)}{\delta_k}\, \psi_k^{1/p} \int_R h\Phi_t(h)\, dh \\
&= \sum_{k=2}^{s/2} \frac{\omega(\delta_k)}{\delta_k}\, \psi_k^{1/p}\, t \int_R h\Phi(h)\, dh \\
&\leqslant C\delta_s \psi_s^{1/p} \sum_{k=2}^{s/2} 2^{k-s} \frac{\omega(\delta_s)}{\delta_s} \\
&\leqslant C\delta_s s^2 \frac{\omega(\delta_s)}{\delta_s} \sum_{k=2}^{s/2} 2^{k-s} = o(\omega(\delta_s)),
\end{aligned}$$

thus (31) is established. Now let us turn to $II(x)$. Obviously,

$$\Phi_t(x)=\frac{1}{t}\,\Phi\left(\frac{x}{t}\right)\geqslant \frac{2\Phi(1)}{2t}\,\chi_{[0,1]}\left(\frac{|x|}{t}\right)\equiv 2\Phi(1)\, K_t(x),$$

$$K(x)=\frac{1}{2}\,\chi_{[0,1]}(|x|).$$

Direct calculations show that

$$K_t * f_s(x) = \tfrac{1}{2}\,\omega(\delta_s)\,\psi_s^{1/p}, \qquad x \in [\alpha_s + \delta_s; \beta_s - \delta_s],$$

thus

$$T_{m_t} f_s(x) \equiv \Phi_t * f_s(x) \geqslant \Phi(1)\,\omega(\delta_s)\,\psi_s^{1/p} \qquad \text{for} \quad x \in [\alpha_s + \delta_s; \beta_s - \delta_s]. \tag{33}$$

Since $E^*_{\xi_m} = \bigcup_{s \in \mathscr{L}_m} E^\tau_s$, where

$$E^\tau_s \equiv \left\{ x \in [\alpha_s + \delta_s; \beta_s - \delta_s] : \operatorname{dist}(x, \mathscr{I}_s) \leqslant \frac{\Phi(1)}{2}\,\delta_s \right\},$$

we obtain for all $x \in E^\tau_s$ that

$$f_s(x) = \psi_s^{1/p}\,\frac{\omega(\delta_s)}{\delta_s}\,\operatorname{dist}(x, \mathscr{I}_s) \leqslant \psi_s^{1/p}\,\frac{\omega(\delta_s)}{\delta_s}\,\frac{\Phi(1)}{2}\,\delta_s = \frac{\Phi(1)}{2}\,\psi_s^{1/p}\,\omega(\delta_s)$$

and therefore, combining the two estimates,

$$T_{m_{\delta_s}} f_s(x) - f_s(x) \geqslant \frac{\Phi(1)}{2}\,\psi_s^{1/p}\,\omega(\delta_s) = \frac{\Phi(1)}{2}\,w(\delta_s), \qquad x \in E^\tau_s. \tag{34}$$

Since every point from $\limsup_m E^*_{\xi_m}$ belongs to infinitely many E^τ_s then it follows from (31), (34), and (27) that for almost all $x \in \mathbf{R}$

$$\limsup_{t \to 0+} \frac{|T_{m_t} f(x) - f(x)|}{w(t)} \geqslant \frac{\Phi(1)}{2}.$$

Finally let us show how to modify the proof to avoid the monotonicity of the kernel. Let $\Psi(x) = \Phi(x)\,\chi_{[0,1]}(|x|)$ and Ψ^* denote the non-increasing rearrangement of the function Ψ. Without loss of generality we may assume that $\Psi^*(1) > 0$. If we set $G = \{x : \Psi(x) > \Psi^*(1)\}$ we obviously have that $|G| = 1$. For $0 < t < 1$ set $G_t = \{x : x/t \in G\}$. Then

$$G_t \subset [-t; t] \qquad \text{and} \qquad |G_t| = t.$$

Using the standard notation $\Psi_t(x) = 1/t\Psi(x/t)$ there holds $\Psi_t(x) \leqslant \Phi_t(x)$ and

$$\Psi_t(x) \geqslant \Psi^*(1)/t, \qquad x \in G_t.$$

Then, for arbitrary $x \in [\alpha_s + \delta_s; \beta_s - \delta_s]$, we have

$$\Phi_t * f_s(x) \geqslant \Psi_t * f_s(x) = \int_{-t}^{t} \Psi_t(y)\, f_s(x-y)\, dy$$

$$\geqslant \int_{G_t} \Psi_t(y)\, f_s(x-y)\, dy \geqslant \frac{\Psi^*(1)}{t} \int_{G_t} f_s(x-y)\, dy.$$

Obviously $x - y \in [x-t; x+t]$ for any $y \in G_t$. Hence

$$\Phi_t * f_s(x) \geqslant \frac{\Psi^*(1)}{t} \inf \int_e f_s(y)\, dy,$$

where the infimum is taken over all measurable sets $e \subset [x-t; x+t]$ with $|e| = t$. Now let us introduce the function

$$\phi_s(x) = \frac{w(\delta_s)}{\delta_s} (\delta_s - |x|)\, \chi_{[0,1]}(|x|/\delta_s).$$

It is clear that f_s is the "finite" periodic extension of ϕ_s, that $[x-t; x+t] \subset \operatorname{supp}(f_s)$ has length $2t$ and coincides with the length of $\operatorname{supp}(\phi_s)$ (in some sense "one period" of f_s), thus (see [1, pp. 44–46])

$$\inf \int_e f_s(y)\, dy = \int_{-t}^{t} \phi_s(y)\, dy - \sup_{|E| = t} \int_E \phi_s(y)\, dy$$

$$= \int_0^{2t} \phi_s^*(y)\, dy - \int_0^{t} \phi_s^*(y)\, dy$$

$$= \int_t^{2t} \phi_s^*(y)\, dy = \frac{w(\delta_s)\, \delta_s}{4}.$$

Hence

$$\Phi_t * f_s(x) \geqslant \frac{\Psi^*(1)}{t} \frac{w(\delta_s)\, \delta_s}{4} = \frac{\Psi^*(1)}{4} w(t)$$

since in our case $t = \delta_s$. This estimate can be used instead of (33) (observe $w(\delta_s) = \psi_s^{1/p} \omega(\delta_s)$). Now change the definition of the set E_s by setting

$$\tilde{E}_s = \bigcup_{v=1}^{r_s - 1} \left[a_s + v2\delta_s - \frac{\Psi^*(1)}{8} \delta_s,\, a_s + v2\delta_s + \frac{\Psi^*(1)}{8} \delta_s \right].$$

This implies corresponding changes in the definition of the set E_s^τ,

$$\tilde{E}_s^\tau \equiv \left\{ x \in [\alpha_s + \delta_s; \beta_s - \delta_s] : \operatorname{dist}(x, \mathcal{J}_s) \leqslant \frac{\Psi^*(1)}{8} \delta_s \right\},$$

and for $x \in \tilde{E}_s^\tau$ there holds

$$f_s(x) \leqslant \frac{\Psi^*(1)}{8} \psi_s^{1/p} \omega(\delta_s).$$

Hence

$$T_{m_{\delta_s}} f_s(x) - f_s(x) \geqslant \frac{\Psi^*(1)}{8} w(\delta_s) = \frac{\Psi^*(1)}{8} w(t)$$

and Theorem 1 is proved.

3. PROOF OF THEOREM 2

This example is in the spirit of the previous one; at the same time the corresponding results of Soljanik [8, 9] for the torus have to be carried over to the real line. Observe that the construction of Soljanik's example $F = \sum F_k$ as a series of analytic functions F_k is technically quite complicated with a long derivation. The reason is to be seen there in the fact that when estimating $F(x+it) - F(x)$ for $t \to 0+$ at a point x two F_k's may have comparable contributions with different signs. The application of the Calderón lemma allows us to separate the influence of the F_k's at x yielding a substantial simplification.

As in Section 2 it is sufficient to prove the existence of some $F \in H_p^\omega(\mathbf{R}_+^2)$ with

$$\limsup_{t \to 0+} \frac{|F(x+it) - F(x)|}{w(t)} > 0 \qquad \text{a.e. on } \mathbf{R} \tag{35}$$

and to assume that (19) is true.

Suppose that the numbers $\{\delta_k\}$ are given by (2) and that q is a fixed positive integer which will be specified later. Take $\mathcal{K}$ as in (23). Define

$$r_k = \max\{m \in \mathbf{Z}_+ : qm\delta_k \leqslant 1/\psi_k\}, \qquad k = 1, 2, \ldots \tag{36}$$

and observe that $\psi_k^{1/p}/r_k \to 0$, in particular $r_k^{-1} \to 0$, for $k \to \infty$ since by [6] $2\delta_{k+1} \leqslant \delta_k$. Define for $k \geqslant 2$ intervals $I_k = (\alpha_k; \beta_k] \equiv (a_k - \delta_k \psi_k^{1/p}; b_k + \delta_k \psi_k^{1/p}]$, where $b_k - a_k = q r_k \delta_k$, in the following way: Set $\alpha_2 = 0$ and $\alpha_{k+1} = \beta_k$ if $\beta_k < 1$ and $\alpha_{k+1} = 0$ otherwise. Let $s_m \uparrow \infty$ be such that $\alpha_{s_m} = 0$ and consider

$$E_k = \bigcup_{v=1}^{r_k - 1} [a_k + (vq-1)\,\delta_k; a_k + (vq+1)\,\delta_k].$$

Then, by (24),

$$|E_k| = 2(r_k - 1)\,\delta_k, \qquad \sum_{k \in \mathscr{K}} |E_k| = \infty.$$

Let $\mathscr{L}$, $\mathscr{L}_m$, and E_m^* be given by (25) and assume (26) to hold. Then, again by the Calderón lemma, there exist numbers ξ_m such that (27) is true.

Denote by τ_m the translation $\tau_m(\cdot) \equiv (\cdot - \xi_m)$ and define

$$I_k^\tau \equiv \tau_m(I_k), \qquad s_{2m} \leqslant k < s_{2m+1}.$$

Since now the distribution of I_k^τ is fixed we may denote it again by the same letters, i.e. assume without loss of generality that they are still in the original positions, so $I_k^\tau = (\alpha_k; \beta_k]$. If for $x \in \mathbf{R}$ we now introduce $\mathscr{K}_x = \{k \in \mathscr{L} : I_k^\tau \ni x\}$ we again have the important property (28).

Let us define a sequence of complex numbers $\{z_{j,k}\}_{j=1}^{r_k}$ by

$$z_{j,k} = a_k + jq\delta_k - i\delta_k, \qquad \text{so} \quad \Re z_{j,k} = a_k + jq\delta_k \tag{37}$$

and for every $k \in \mathscr{K}$ set

$$F_k(z) = w(\delta_k) \sum_{j=1}^{r_k} \left(\frac{\delta_k}{z_{j,k} - z} \right)^2, \qquad z \in \mathbf{C}, \quad \Im z > -\delta_k.$$

We note that F_k restricted to the real line is bounded,

$$\|F_k\|_\infty \leqslant C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^2}{(jq\delta_k)^2} \leqslant C_p w(\delta_k) \tag{38}$$

and, therefore,

$$\|F_k\|_p \leqslant \|F_k\|_\infty \left(\int_{x \in 3I_k} dx \right)^{1/p} + \left(\int_{x \notin 3I_k} |F_k(x)|^p \, dx \right)^{1/p} \leqslant C_p \omega(\delta_k) + \cdots$$

since $|I_k| \approx \psi_k^{-1} \approx r_k \delta_k$ (see the definition (19) of ψ_k). Also

$$\left(\int_{x \notin 3I_k} |F_k(x)|^p \, dx\right)^{1/p} \leqslant w(\delta_k)\, \delta_k^2 \sum_{j=1}^{r_k} \left(\int_{x \notin 3I_k} \frac{dx}{|z_{j,k} - x|^{2p}}\right)^{1/p}$$

$$\leqslant C_p w(\delta_k)\, r_k \delta_k^2 \left(\int_{x \geqslant |I_k|} \frac{dx}{(\delta_k^2 + x^2)^p}\right)^{1/p}$$

$$\leqslant C_p \omega(\delta_k)\, \psi_k^{1/p} r_k \delta_k^2\, |I_k|^{1/p-2}.$$

$$\leqslant C_p \omega(\delta_k)/r_k = o(\omega(\delta_k)),$$

hence

$$\|F_k\|_p \leqslant C_p \omega(\delta_k). \tag{39}$$

Further

$$|F'_k(x)| \leqslant C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^2}{(jq\delta_k)^3} = C_p \frac{w(\delta_k)}{q^3 \delta_k} \sum_{j=1}^{\infty} j^{-3} = C_{p,q} \frac{w(\delta_k)}{\delta_k}$$

whence

$$\|F'_k\|_\infty \leqslant C_{p,q} \frac{w(\delta_k)}{\delta_k}. \tag{40}$$

If $x \notin I_k$ then

$$|F_k(x)| \leqslant C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^2}{(\delta_k + jq\delta_k + \delta_k \psi_k^{1/p})^2}$$

$$\leqslant C_p \frac{w(\delta_k)}{q^2} \sum_{j=1}^{\infty} (jq + \psi_k^{1/p})^{-2}$$

$$\leqslant C_{p,q} w(\delta_k)(\psi_k^{1/p})^{-1} = C_{p,q} \omega(\delta_k)$$

and

$$\|F_k \chi_{(I_k)^c}\|_\infty \leqslant C_{p,q} \omega(\delta_k). \tag{41}$$

Also

$$|F'_k(x)| \leqslant C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^2}{(\delta_k + jq\delta_k + \delta_k \psi_k^{1/p})^3}$$

$$\leqslant C_p \frac{w(\delta_k)}{q^3 \delta_k} \sum_{j=1}^{\infty} (jq + \psi_k^{1/p})^{-3}$$

$$\leqslant C_{p,q} \frac{w(\delta_k)}{\delta_k} (\psi_k^{1/p})^{-2} \leqslant C_{p,q} \frac{\omega(\delta_k)}{\delta_k}$$

and

$$\|F'_k \chi_{(I_k)^c}\|_\infty \leqslant C_{p,q} \frac{\omega(\delta_k)}{\delta_k}. \tag{42}$$

Now define $F = \sum_{k \in \mathscr{L}} F_k$. In view of (2) and (19) the estimates (38) and (39) imply that F is bounded analytic in $\mathbf{R}^2_+$ and belongs to $H^p(\mathbf{R}^2_+)$. We show that $F \in H^\omega_p(\mathbf{R}^2_+)$.

Choose $\delta_{s+1} < h \leqslant \delta_s$. Then

$$\begin{aligned}\omega(F, h)_p &\leqslant \sum_{k \leqslant s,\, k \in \mathscr{L}} \omega(F_k, h)_p + 2 \sum_{k > s,\, k \in \mathscr{L}} \|F_k\|_p \\ &\leqslant \sum_{k \leqslant s,\, k \in \mathscr{L}} \omega(F_k, h)_p + C_p \omega(\delta_s)\end{aligned}$$

by (2) and (39). Now

$$\begin{aligned}\|F_k(x+h) - F_k(x)\|_p &\leqslant \left(\int_{x \in 5I_k} |F_k(x+h) - F_k(x)|^p \right)^{1/p} \\ &\quad + \left(\int_{x \notin 5I_k} |F_k(x+h) - F_k(x)|^p \right)^{1/p} \\ &\equiv I_1 + I_2.\end{aligned}$$

By (40)

$$I_1 \leqslant h\, \|F'_k\|_\infty \left(\int_{x \in 5I_k} dx \right)^{1/p} \leqslant C_p h w(\delta_k)\, \delta_k^{-1} \psi_k^{-1/p} \leqslant C_p h \omega(\delta_k)\, \delta_k^{-1}.$$

Further,

$$I_2 \leqslant C_p h w(\delta_k)\, \delta_k^2 \sum_{j=1}^{r_k} \left(\int_{x \notin 5I_k} \frac{dx}{|z_{j,k} - x - \xi_j|^{3p}} \right)^{1/p}$$

with some $0 \leqslant \xi_j < h \leqslant \delta_k$. Since $x + \xi_j \notin 3I_k$ we have

$$\left(\int_{x \notin 5I_k} \frac{dx}{|z_{j,k} - x - \xi_j|^{3p}} \right)^{1/p} \leqslant C_p \left(\int_{|x| \geqslant |I_k|} \frac{dx}{x^{3p}} \right)^{1/p} \leqslant C_p \psi_k^{3-1/p}.$$

Hence

$$I_2 \leqslant C_p w(\delta_k)\, \delta_k^2 h r_k \psi_k^{3-1/p} = C_p h \frac{\omega(\delta_k)}{\delta_k} \frac{1}{r_k^2} \leqslant C_p h \frac{\omega(\delta_k)}{\delta_k}$$

and thus $\omega(F_k, h)_p \leqslant C_p h\omega(\delta_k)\,\delta_k^{-1}$. Since

$$\sum_{k\leqslant s,\, k\in\mathcal{L}} \omega(F_k, h)_p \leqslant C_{p,q} h \sum_{k\leqslant s} \omega(\delta_k)\,\delta_k^{-1} \leqslant C_{p,q} h\omega(\delta_s)\,\delta_s^{-1} \leqslant C_{p,q}\omega(h)$$

we obtain

$$\omega(F, h)_p \leqslant C_{p,q}\omega(h),$$

i.e., $F\in H_p^{\omega}(\mathbf{R}_+^2)$.

Next we examine the behavior of $F(x+it)-F(x)$. Take $t=\delta_s$ with $s\in\mathcal{L}$. Then

$$|F(x+it)-F(x)| \geqslant |F_s(x+it)-F_s(x))| - \sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)| - \sum_{k>s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)|.$$

We discuss the contributions of these terms. First we have

$$\sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)| = \sum_{k<s,\, k\in\mathcal{K}_x} |F_k(x+it)-F_k(x)| + \sum_{k<s,\, k\in\mathcal{K}_x^c} |F_k(x+it)-F_k(x)| \equiv \Sigma_1+\Sigma_2$$

where again $\mathcal{K}_x^c=\mathcal{L}\backslash\mathcal{K}_x$. Then, by (28) and (23),

$$\Sigma_1 \leqslant \sum_{k<s,\, k\in\mathcal{K}_x} \delta_s \|F_k'\|_\infty \leqslant C_{p,q}\delta_s \sum_{k\leqslant s/2} \omega(\delta_k)\,\psi_k^{1/p}\delta_k^{-1} \leqslant C_{p,q}\delta_s \frac{\omega(\delta_s)}{\delta_s} s^2 \sum_{k\leqslant s/2} 2^{k-s} \leqslant C_{p,q} s^2 2^{-s/2}\omega(\delta_s) = o(\omega(\delta_s))$$

and

$$\Sigma_2 \leqslant \sum_{k<s,\, k\in\mathcal{L}} \delta_s \|F_k'\chi_{(I_k)^c}\|_\infty \leqslant C_{p,q}\delta_s \sum_{k<s} \frac{\omega(\delta_k)}{\delta_k} \leqslant C_{p,q}\omega(\delta_s).$$

Combining these two estimates we have for sufficiently large s

$$\sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)| \leqslant C_{p,q}\omega(\delta_s). \tag{43}$$

Analogously, we decompose

$$\sum_{k>s,\, k\in\mathscr{L}} |F_k(x+it)-F_k(x)| = \sum_{k>s,\, k\in\mathscr{K}_x} |F_k(x+it)-F_k(x)| + \sum_{k>s,\, k\in\mathscr{K}_x^c} |F_k(x+it)-F_k(x)| \equiv \Sigma^1+\Sigma^2.$$

Then, by (28), (23), and (38),

$$\Sigma^1 \leqslant 2\sum_{k>s,\, k\in\mathscr{K}_x} \|F_k\|_\infty \leqslant C_{p,q}\sum_{k\geqslant 2s} w(\delta_k) \leqslant C_{p,q}\omega(\delta_s)\sum_{k\geqslant 2s} 2^{s-k}k^{2/p} \leqslant C_{p,q}\omega(\delta_s)\, s^2 2^{-s} = o(\omega(\delta_s))$$

and by (41)

$$\Sigma^2 \leqslant \sum_{k>s,\, k\in\mathscr{K}_x^c} \|F_k\chi_{cI_k}\|_\infty \leqslant C_{p,q}\sum_{k>s}\omega(\delta_k) \leqslant C_{p,q}\omega(\delta_s).$$

Thus, for sufficiently large s,

$$\sum_{k>s,\, k\in\mathscr{L}} |F_k(x+it)-F_k(x)| \leqslant C_{p,q}\omega(\delta_s) \tag{44}$$

(recall that we have set $t=\delta_s,\ s\in\mathscr{L}$). Therefore, as a consequence of (43) and (44), we have that

$$|F(x+it)-F(x)| \geqslant |F_s(x+it)-F_s(x)| + O(\omega(\delta_s)). \tag{45}$$

For $x\in E_s$ with $|\Re z_{j,s}-x|\leqslant\delta_s$ there follows

$$|F_s(x+it)-F_s(x))| \geqslant w(\delta_s)\left(\frac{\delta_s^2}{|z_{j,s}-x|^2} - \frac{\delta_s^2}{|z_{j,s}-x-it|^2} - \sum_{|j-n|\geqslant 1}\frac{\delta_s^2}{|z_{n,s}-x|^2} - \sum_{|j-n|\geqslant 1}\frac{\delta_s^2}{|z_{n,s}-x-it|^2}\right) \equiv w(\delta_s)(A-B-C-D).$$

But it is easy to see that $A-B\geqslant 1/4$ and $D\leqslant C$. Finally, by (37),

$$C \leqslant \sum_{|j-n|\geqslant 1}\frac{\delta_s^2}{|\Re(z_{n,s}-x)|^2} \leqslant \sum_{|j-n|\geqslant 1}\frac{\delta_s^2}{(|\Re z_{n,s}-\Re z_{j,s}|-|\Re z_{j,s}-x|)^2} \leqslant \sum_{|j-n|\geqslant 1}\frac{\delta_s^2}{(q\,|n-j|\,\delta_s-\delta_s)^2} \leqslant 2\sum_{n=1}^{\infty}\frac{\delta_s^2}{|(qn-1)\,\delta_s|^2} \leqslant cq^{-2}.$$

Now choose q such that $cq^{-2} \leqslant 1/16$. Then

$$|F_s(x+it) - F_s(x)| \geqslant w(\delta_s)/8$$

which together with (45) implies that for the given x

$$|F(x+it) - F(x))| \geqslant \tfrac{1}{8} w(\delta_s) + O(\omega(\delta_s)) = \tfrac{1}{8} w(\delta_s) + o(w(\delta_s))$$

from which (18) follows and Theorem 2 is proved.

4. PROOF OF THEOREM 3

The proof of Part (a) is a simple consequence of the more general theorem from [11] which for the convenience of the reader we formulate in the instance of the Abel–Poisson means.

THEOREM B. *Let $\omega(t)$ be a modulus of continuity and define θ_k such that $\omega(\theta_k) = 2^{-k}$. Then, for every nondecreasing function $w(t)$ with $\omega(t)/w(t)$ nondecreasing and*

$$\Sigma \equiv \sum_{j=1}^{\infty} 2^{-j} c_j \left(\sum_{k=1}^{\infty} \frac{\omega(\theta_k)}{w(2^{-j}\theta_k)} \right) < \infty \tag{46}$$

and for every function $f \in H_1^{\omega}(\mathbf{R})$, the estimate (15) *holds, where c_j are any positive numbers such that $\sum_{j=1}^{\infty} 1/c_j$ converges.*

Since without loss of generality we may assume that $\omega(2t) \leqslant Q\omega(t)$ uniformly in $0 < t < 1$ for some $Q < 2$, (14) clearly implies (46) because

$$\Sigma \leqslant \sum_{j=1}^{\infty} 2^{-j} c_j \left(\sum_{k=1}^{\infty} \frac{Q^j \omega(2^{-j}\theta_k)}{w(2^{-j}\theta_k)} \right) \leqslant \sum_{j=1}^{\infty} Q^j 2^{-j} c_j \left(\sum_{k=1}^{\infty} \frac{\omega(\theta_k)}{w(\theta_k)} \right)$$

and Part (a) is proved.

Let us turn to the proof of Part (b). It is sufficient to prove (17) with positive right sides (see the argument at the beginning of the proof of Theorem 1). Set $q \equiv Q/2 < 1$ and recall that $\omega(\theta_k) = 2^{-k}$. Obviously $2\theta_{k+1} \leqslant \theta_k$ for arbitrary $\omega(t)$, so

$$\frac{\omega(\theta_k)}{\theta_k} \leqslant \frac{\omega(2\theta_{k+1})}{2\theta_{k+1}} \leqslant q \frac{\omega(\theta_{k+1})}{\theta_{k+1}}$$

yielding

$$\sum_{k=1}^{m} \frac{\omega(\theta_k)}{\theta_k} \leqslant C_q \frac{\omega(\theta_m)}{\theta_m}, \tag{47}$$

$$\frac{\omega(\theta_k)}{\theta_k} \leqslant q^k \frac{\omega(\theta_{2k})}{\theta_{2k}}. \tag{48}$$

Further, we repeat the construction of the function f from the proof to Theorem 1 with the only difference that we replace δ_k by θ_k. We again show that $f \in H_1^{\omega}(\mathbf{R})$. For $\theta_{m+1} < h \leqslant \theta_k$ by (47) and (48) we have

$$\begin{aligned}\omega(f, h)_1 &\leqslant \sum_{k \leqslant m,\, k \in \mathcal{L}} \omega(f_k; h)_1 + 2 \sum_{k > m,\, k \in \mathcal{L}}^{\infty} \|f_k\|_1 \\ &\leqslant C \sum_{k \leqslant m,\, k \in \mathcal{L}} \omega(f_k; h)_\infty \psi_k^{-1} + C \sum_{k=m+1}^{\infty} \omega(\theta_k) \\ &\leqslant Ch \sum_{k=1}^{m} \frac{\omega(\theta_k)}{\theta_k} + C\omega(\theta_{m+1}) \\ &\leqslant C_q \left(h \frac{\omega(\theta_m)}{\theta_m} + \omega(\theta_{m+1}) \right) \\ &\leqslant C_q \omega(h),\end{aligned}$$

hence $f \in H_1^{\omega}(\mathbf{R})$.

Now choose $t = \theta_s$ and estimate $III(x)$, $I(x)$ and $II(x)$ being generated in the same way as in the proof of Theorem 1. Then

$$\begin{aligned}|III(x)| &\leqslant \sum_{k > s,\, k \in \mathcal{K}_x} \int_R |f_k(x+h) - f_k(x)|\, P_t(h)\, dh \\ &\leqslant \sum_{k \geqslant 2s} 2\omega(\theta_k)\, \psi_k \leqslant C2\omega(\theta_{2s})\, \psi_{2s} \leqslant C2^{-2s} s^2 = o(\omega(t)).\end{aligned} \tag{49}$$

Concerning the estimate of $I(x)$ we have

$$\begin{aligned}|I(x)| &\leqslant \sum_{k < s,\, k \in \mathcal{K}_x} \left(\int_{|h| \leqslant t2^{2s}} + \int_{|h| \geqslant t2^{2s}} \right) |f_k(x+h) - f_k(x)|\, P_t(h)\, dh \\ &\equiv \Sigma_1 + \Sigma_2.\end{aligned}$$

By (47) and (48) there follows

$$\begin{aligned}\Sigma_1 &\leqslant t \sum_{k<s/2} \frac{\omega(\theta_k)\,\psi_k}{\theta_k} \int_{|h|\leqslant t2^{2s}} \frac{h}{t} P_t(h)\,dh \\ &\leqslant Cts^2 \sum_{k<s/2} \frac{\omega(\theta_k)}{\theta_k} \int_{|h|\leqslant 2^{2s}} \frac{h\,dh}{1+h^2} \leqslant Cts^3 \sum_{k<s/2} \frac{\omega(\theta_k)}{\theta_k} \\ &\leqslant Cts^3 \frac{\omega(\theta_{s/2})}{\theta_{s/2}} \leqslant Cts^3 q^{s/2} \frac{\omega(\theta_s)}{\theta_s} \\ &= Cs^3 q^{s/2} \omega(\theta_s) = o(\omega(t)), \end{aligned} \tag{50}$$

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{k<s/2} \omega(\theta_k)\,\psi_k \int_{|h|\geqslant t2^{2s}} P_t(h)\,dh \\ &\leqslant Cs^2 \int_{|h|\geqslant 2^{2s}} \frac{1}{1+h^2}\,dh \sum_{k<s/2} \omega(\theta_k) \leqslant Cs^2 2^{-2s} = o(\omega(t)). \end{aligned} \tag{51}$$

Summarizing, by (49)–(51) there holds

$$I(x) + III(x) = o(\omega(t)), \qquad t \to 0+. \tag{52}$$

Finally we note that the estimate $II(x)$ is the same as that in Theorem 1:

$$P_t f_s(x) \geqslant P(1)\,\omega(\theta_s)\,\psi_s, \qquad x \in [\alpha_s + \theta_s; \beta_s - \theta_s],$$

which implies that

$$P_t f_s(x) - f_s(x) \geqslant \frac{P(1)}{2} \psi_s \omega(\theta_s) = \frac{P(1)}{2} w(\theta_s), \qquad x \in E_s^\tau. \tag{53}$$

Thus, a combination of (52) and (53) yields

$$\limsup_{t\to 0+} \frac{|P_t f(x) - f(x)|}{w(t)} \geqslant \frac{P(1)}{2} \qquad \text{a.e. on } \mathbf{R}$$

and Theorem 3 is completely proved.

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REFERENCES

1. C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.
2. R. A. DeVore, S. D. Riemenschneider, and R. L. Sharpley, Weak interpolation in Banach spaces, *J. Funct. Anal.* **33** (1979), 58–94.
3. Ch. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
4. B. I. Golubov, On Abel–Poisson type and Riesz means, *Anal. Math.* **7** (1981), 161–184.
5. M. de Guzman, "Real Variable Methods in Fourier Analysis," North-Holland, Amsterdam, 1981.
6. K. I. Oskolkov, Approximative properties of summable function on sets of full measure, *Mat. Sb.* **103**, No. 4 (145) (1977), 563–589; English translation, *Math. USSR-Sb.* **32** (1977), 489–514.
7. K. I. Oskolkov, On Luzin's *C*-property for a conjugate function, *Trudy Mat. Inst. Steklov.* **164** (1983), 124–135; English translation, *Proc. Steklov Inst. Math.* **164** (1985), 141–153.
8. A. A. Soljanik, Rate of approximation of H^p functions by Abel–Poisson means almost everywhere, *Soobshch. Akad. Nauk Gruzii.* **125**, No. 2 (1987), 245–248. [In Russian]
9. A. A. Soljanik, "Approximation of Functions from Hardy Classes," Ph.D. Thesis, Odessa, 1986.
10. E. M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals," Princeton Univ. Press, Princeton, NJ, 1993.
11. A. Stokolos and W. Trebels, On the rate of almost everywhere convergence of certain classical integral means, *J. Approx. Theory* **98** (1999), 203–222.